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# STS-graphs of perfect codes mod kernel☆

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## Abstract

We show that a perfect 1-error-correcting code $C$ is 'foldable' over its kernel via the Steiner triple systems associated to its codewords. The resulting 'folding' produces a graph invariant for $C$ via Pasch configurations and tensors. Moreover, the invariant is complete for Vasil'ev codes of length 15, showing among other things the existence of nonadditive propelinear 1-perfect codes, and allowing to visualize a propelinear code by means of the commutative group formed by its classes mod kernel, as well as to generalize the notion of a propelinear code by extending the involved composition of permutations to a more general group product.


*Keywords:* Perfect 1-error-correcting code; Steiner triple system; Graph invariant

## 1. Introduction

In this paper, terminology from graph theory is used as in [2,22]; from coding theory as in [8,13,16]; and from Steiner triple systems as in [5].

Let $1\leqslant n \in \mathbb{Z}$. The *n-cube* $Q_n$ is the simple graph with vertex set $\{0,1\}^n = GF_2^n$ and an edge between each two vertices that differ in exactly one coordinate. Then, a 1-*perfect code* (or perfect 1-error-correcting code) $C = C_r$ of length $n = 2^r - 1$ can be seen as an independent vertex set of $Q_n$ such that each vertex of $Q_n \backslash C$ is neighbor of exactly one

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vertex of $C$, where $0 < r \in \mathbb{Z}$. It follows that $C$ has $2^{n-r}$ vertices and minimum Hamming distance 3. For every $r$ as above, at least a $C_r$ exists, namely the linear one. This $C_r$ is unique for $r < 4$; for $r \geqslant 4$, many nonlinear $C_r$'s exist [21,14,15,7,17,20].

The *kernel* $K = Ker(C)$ of $C$ is the set formed by all the vertices $v \in Q_n$ such that $v + C = C$ [15]. If $0 \in C$, then $K \subseteq C$; in this case, $K$ is the intersection of all maximal linear subcodes contained in $C$.

Three invariants for 1-perfect codes $C$ have been used to classifying such codes: the dimension of the kernel (see [15]); the rank, (i.e. the dimension of the linear subspace spanned by $C$, see [15]); and the STS-graph $H(C)$ of [6]. A new invariant based on any linear subspace $L \subseteq K$, is defined in Section 2 and denoted $H_L(C)$.

In what follows, the concepts of effectiveness and completeness of an invariant are taken as in [4, p. 57] and used to test the effectiveness of $H_K(C)$ for Vasil'ev codes $C$ of length 15; the outcome, presented in Theorem 1 is that $H_K(C)$ constitutes a complete invariant for those codes.

Let $I_n = \{1, \ldots, n\}$. A 1-perfect code $C \subset Q_n$ is said to be *prelinear* if $0 \in C$ and a set $\Pi = \{\pi_w | w \in C\}$ of permutations of $I_n$ exists such that for every $u \in C, u * v = u + \pi_u(v) \in C$ if and only if $v \in C$. A prelinear 1-perfect code $C$ is *weakly propelinear* if $\Pi$ is a group under a product $\circ'$, (not necessarily the usual composition $\circ$), such that for every $u, v \in C$, the codeword $w = u * v = u + \pi_u(v)$ satisfies $\pi_u \circ' \pi_v = \pi_w \in \Pi$. If $\circ' = \circ$, then $C$ is *propelinear*, in which case $(C, *)$ is a group (see [18]).

Borges and Rifà [4] found that a propelinear 1-perfect code is additive if it is of the form $\mathbb{Z}_2^p \times \mathbb{Z}_4^q$. However, a nonadditive propelinear 1-perfect code was not known that could not be provided with an additive structure. (The linear code $C_4$ admits a nonadditive propelinear 1-perfect structure of the form $\mathbb{Z}_2^3 \times \mathbb{Z}_4^2$, see [4]. Also, Preparata codes are nonadditive propelinear and Preparata-like $Z_4$-codes are additive propelinear, but neither is 1-perfect, see [19]. In addition, a propelinear code is always homogeneous, see [3]).

**Theorem 1.** *Let $C$ be a Vasil'ev code of length* 15. *Then $H_K(C)$ is a complete invariant for $C$. Moreover, $H_K(C)$ allows to distinguish: two classes of nonadditive propelinear 1-perfect codes; one of nonlinear Vasil'ev additive codes; one of nonpropelinear weakly propelinear codes; and two of nonprelinear homogeneous codes. Furthermore, $H_K(C)$ presents any weakly propelinear code $C$ as a commutative-group relation between classes mod $Ker(C)$ via $*$ and $\circ'$. This relation yields a group $(C/Ker(C), *')$ which lifts to the group $(C, *)$ if and only if $C$ is propelinear.*

Theorem 3 states that all 1-perfect codes $C$ are 'foldable' over the linear subspaces $L \subseteq K$ via the Steiner triple systems associated to the codewords of $C$. This is necessary for proving Theorem 1, because the resulting 'folding' $H_L(C)$ shows a convenient view of the combinatorial and geometric properties of Vasil'ev codes $C$ of length 15. More specifically, let $J$ be the linear subspace $L \subseteq K$ generated by the seven Steiner triples that share coordinate 15 in any codeword (of the linear code, for example). Then, for any Vasil'ev code $C$ of length 15, the invariant graph $H_J(C)$ can be expressed via a tensor product of two binary 8-vectors involving the Pasch configurations associated with the mentioned Steiner triple systems. Sections 3 and 4 constitute our way to show that such product depends solely on the 19 equivalence classes obtained by Hergert in [9,10], which is stated as Theorem 5. Based on this, Section 5 shows that the invariant $H_K(C)$ distinguishes completely between each pair of equivalence classes of Vasil'ev codes in Hergert's classification, as claimed in Theorem 1. Section 6 establishes the rest of the claims of Theorem 1 by showing that the code classes cited in the statement are represented, in the notation of Section 3, respectively in $\{V_2, V_5\}$; $\{V_1\}$; $\{V_6\}$; and $\{V_{13}, V_{15}\}$; where $V_1$ is the only additive Vasil'ev code (due to [3]) and the only existing code for which the last sentence of the theorem occurs affirmatively.

An additional advantage of our approach is that the considered 'folding' makes computability of the invariant graphs $H_L(C)$ much quicker than that of $H(C)$ (see Section 2).

Lemma 2 and Theorem 3 yield Corollaries 8 and 9. The first compares $H_K(C)$ to $H(C)$. The second establishes the conjecture that at most two iterations are needed to obtain $H(C)$ via its inductive definition (see [6]).

## 2. Foldability and STS-graphs mod kernel

The minimum-distance graph $M(C)$ of a 1-perfect code $C$ has $C$ as its vertex set and exactly one edge between every two vertices $v, w \in C$ such that the Hamming distance between them is $d(v, w) = 3$. Each edge $vw$ of $M(C)$ is labeled by the triple of coordinate indices in $I_n$ making up $d(v, w) = 3$. The labels on the edges of $M(C)$ incident to any vertex $v$ constitute a Steiner triple system $S(C, v)$ formed by $n(n-1)/6$ triples (see [16]).

Given an edge $vw$ of $M(C)$, its labelling triple is referred to as $s(vw)$ and called the STS-label on $vw$, where STS stands for Steiner triple system. Also, we label each codeword $v$ of $C$ by using the equivalence class $\mathscr{S}[v]$ of Steiner triple systems on $n$ elements, (called for short STS($n$)-type $\mathscr{S}[v]$), corresponding to $S(C, v)$. The STS-graph of $C$ is $M(C)$ together with all these vertex and edge labels.

Let $L$ be a linear subspace of $K$, where $L$ may equal $K$. $L$ partitions $C$ into subsets $v + L$, where $w \in C$ belongs to $v + L$ if and only if $v - w \in L$. These subsets $v + L$ are the classes of $C$ mod $L$, and they are considered as a quotient set $C/L$.

**Lemma 2.** *Each class $v + L \in C/L$ can be assigned a unique Steiner triple system $S(C, v)$ in the sense that $S(C, v) = S(C, u)$ for each $u \in v + L$.*

**Proof.** This is an extension of Lemma 4.24 of [3], where $L = K$. □

If no confusion arises, every codeword of $C$ will be written either as an $n$-tuple or as its support in $I_n$. For example, given two codewords $u, v$ of $C$ such that $d(u, v) = 3$, we will write $s(uv) = v - u$. $C$ is *foldable over* $L$ if for every two classes $u + L$ and $v + L$ of $C$ mod $L$ such that $s(uv) = v - u$ holds that for every $u' \in u + L$ there is a $v' \in v + L$ such that $s(u'v') = v - u$. If $C$ is foldable, we can furnish $C/L$ as a quotient graph $H_L(C)$ of $M(C)$ by setting an edge between each two classes $u + L, v + L$, if $uv$ is an edge of $M(C)$. Moreover, $uv$ admits $s(uv)$ as a well-defined label. Furthermore, $M(C)$ can be recovered from the resulting labeled graph $H_L(C)$. This yields the properties contained in Corollary 4.

**Theorem 3.** *Every* 1*-perfect code* $C$ *is foldable over any linear* $L \subseteq K$.

**Proof.** Let $u + L,\ v + L \in C/L$ be such that $s(u, v) = v - u$. If $x$ in $u + L$, then there exists a codeword $y$ in $v + L$ such that $s(x, y) = y - x = s(u, v) = v - u$. In fact, $y = x + (v - u)$. Now, $y$ is a vector in $v + L$, since $y = x + v - u = (x - u) + v$, where $x - u$ is in $L$. □

A graph covering is a graph map $\phi : G \to H$ for which a nonnegative integer $s$ exists such that the inverse image $\phi^{-1}$ of each vertex and of each edge of $H$ has cardinality $s$.

**Corollary 4.** *If* $C$ *is foldable over a linear subspace* $L$ *of* $K$*, then the quotient map* $C \to C/L$ *can be extended to a graph covering* $M(C) \to H_L(C)$*. Each edge* $e = (v + L)(w + L)$ *of* $H_L(C)$ *can be assigned a well-defined set of triples coincident with the set of STS-labels on the edges of the inverse-image of* $e$*. Moreover, if* $C$ *is foldable over* $K$*, then it is foldable over* $L$*.*

The graph $H_L(C)$ in Corollary 4 is the *STS-graph of* $C$ mod $L$.

Corollary 8 shows that $H_K(C)$ constitutes, as an invariant for $C$, a refinement of $H(C)$, for the vertices of $H(C)$ represent either classes or unions of classes mod $K$. Moreover, if $n = 15$, for which a STS($n$) has exactly 35 triples, the following is a much simpler method of computing $H_K(C)$ or $H_J(C)$ than the one presented in [6] for $H(C)$.

Denote the type of a STS(15) by its associated integer $t \in I_{80}$ in the common-order lists of the 80 existing STS(15) types in [23,12,5,11]. An algorithm in [6] for determining $H(C_4)$ calculates incidentally the STS(15)-types $\mathscr{S}[v]$ associated with the $2^{11} = 2048$ codewords $v \in C_4$. To determine $H_K(C_4)$ or $H_J(C_4)$ we need only to calculate the STS(15)-type of only one codeword per class mod $K$ or mod $J$, respectively. The number of these classes is not larger than $2^{11-7} = 16$ classes for the Vasil'ev codes, which reduces the 2048 calculations in [6] to $\leqslant 16$ here.

A *fragment* $F = \{x_1x_2, x_3x_4, x_5x_6\}$ of a STS(15) $S$, constituted by three ordered pairs $x_1x_2$, $x_3x_4$, $x_5x_6$ in $\{1, \ldots, 15\}$, is the subsystem of $S$ formed by the 4 triples $\{x_1x_3x_5, x_1x_4x_6, x_2x_3x_6, x_2x_4x_5\}$ (see [11]). The order of $x_1x_2$, $x_3x_4$, $x_5x_6$ is irrelevant, since permuting the pairs leaves $F$ unaltered.

In [11], LeVan associated to each STS(15) $S$ the concatenation of: (a) the cardinality $\gamma(S)$ of the family of fragments of $S$; and (b) a nondecreasing 15-tuple $\Gamma(S)$ whose terms are the cardinalities $\gamma_k(S)$ of the subfamilies of those fragments of $S$ containing each $k \in I_{15}$. This determines $S$ uniquely in the collection $\mathscr{S}$ of the STS(15)-types, and thus constitutes an invariant of STS(15)-types.

We consider each $C_4$ as a list of 2048 15-tuples of the alphabet $GF_2$. Each tuple $v$ of $C_4$ determines the STS(15)-type $\mathscr{S}[v]$ represented as the invariant $\gamma(\mathscr{S}[v])(\Gamma(\mathscr{S}[v]))$. For example, Vasil'ev codes have only codewords of STS(15)-type 1 or 2 (see [1,6] and items F–G of Section 4), whose respective invariants $\gamma(\mathscr{S}[v])(\Gamma(\mathscr{S}[v]))$ are:

$$105(42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42), \tag{1}$$

$$73(42, 30, 30, 30, 30, 30, 30, 30, 30, 26, 26, 26, 26, 26, 26). \tag{2}$$

## 3. Vasil'ev codes of length 15

A Vasil'ev code of length 15 is defined as a subset $V$ of $Q_{15}$ of the form

$$V = \{(u, u+v, \pi(u)+g(v)); u \in GF_2^7, v \in LC_3\}, \tag{3}$$

where $LC_3$ is the linear code $C_3$ whose parity-check matrix $H_7$ (taken as in [8]) has as its rows the binary representations of the integers $1, \ldots, 7$ in order, with enough zeros set to the left in order to form 3-tuples; $\pi : GF_2 \to GF_2$ is the parity function given by the weight of 7-tuples taken mod 2 and $g : GF_2^7 \to GF_2$ is any binary function such that $g(0) = 0$.

Hergert [9,10] categorized the Vasil'ev codes of length 15 into 19 equivalence classes. In Section 4, we introduce $H_K(V_k)$ and $H_J(V_k)$, where $V_k$ runs over a set of 19 representative Vasil'ev codes, $(0 \leqslant k \leqslant 18)$, in 1-1 correspondence with the set of Hergert classes via inclusion.

A *V-fragment* is a fragment of the form $\{abc, a'b'c, a'bc', ab'c'\}$, with $t' = 15 - t$ for $t \in \{a, b, c\}$ either contained in $I_7$ or in $I_{14} \backslash I_7$ .

We obtain a deeper view of the combinatorial and geometric properties of the Vasil'ev codes $V_k$ in terms of their local STS(15)s, through items A–H and the resulting Theorem 5. In fact, those local STS(15) split into the seven generators of $J$ plus seven V-fragments covering the remaining $35 - 7 = 28$ triples. Each of these V-fragments has two possible structures containing either a triple of indices of linearly dependent rows of $H_7$ or the triple of their complements mod 15, which we denote respectively $+$ or $-$. Furthermore, each $H_J(V_k)$ is obtained from $K_{8,8}$ by the removal of a 1-factor $F$, so that the vertex and edge STS-labels that $H_J(V_k)$ inherits from $M(V_k)$ are expressible in terms of a tensor product of associated 8-tuples characteristic of each Hergert class up to STS equivalence (see Theorem 5). We will see that the vertex parts of the resulting representation $K_{8,8} - F$ of $H_J(V_k)$ can be taken to be $\{0, \ldots, 7\}$ and $\{8, \ldots, 15\}$, with the 1-factor $F$ formed by the pairs $(j, 15-j)$, for $j \in I_7$. This yields a common disposition for the 19 codes $V_i$.

Let $PG(3, 2)$ be the 3-dimensional binary projective space associated with the parity-check matrix $H_{15}$ of the linear code $V_1$. $PG(2, 3)$ becomes an ordered space by comparing its elements in the decimal notation obtained from their binary vector reading. Then, the common disposition above arises by supplanting Hergert's original ordered set of coordinates by this ordered space. Then, the permutation (8 14)(9 13)(10 12) results in a change of coordinates so that entries 8 to 14 of [9,10] correspond here to entries 14 down to 8, respectively, while the other entries are kept unchanged. This is fundamental for reaching the tensor expressions of items E–H.

We now introduce $V_k$, $(0 \leqslant k \leqslant 18)$, according to the corrected version [10] of [9]:

$$V_k = \{(x, y, f_8(X), f_9(X), f_{10}(X), z, f_{15}(X)); x, z \in GF_2^4, y \in GF_2^3\},$$

where $X = (x, y, z) = (x_1, x_2, x_3, x_4;\ y_5, y_6, y_7;\ z_{11}, z_{12}, z_{13}, z_{14}) \in GF_2^{11}$,

$$\begin{aligned}
f_8(X) &= x_1 + x_2 + x_4 + y_7 + z_{14} + z_{13} + z_{11},\\
f_9(X) &= x_1 + x_3 + x_4 + y_6 + z_{14} + z_{12} + z_{11},\\
f_{10}(X) &= x_2 + x_3 + x_4 + y_5 + z_{13} + z_{12} + z_{11},\\
f_{15}(X) &= x_1 + x_2 + x_3 + x_4 + y_5 + y_6 + y_7 + \overline{g}_k(x, z),
\end{aligned}$$

$\overline{g}_k(x,z)$ is determined by the function $g(v)$ in (3) and $f_8, f_9, f_{10}, f_{15}$ are obtained by replacing $v_t = x_t + z_{15-t}$, $(t \in I_4)$, in the sum of the components of the support $H_k$ of a minimal-weight binary 11-tuple in the basis $\{e_1, \ldots, e_{11}\}$ given by

$$e_1 = v_1v_2, \quad e_2 = v_1v_3, \quad e_3 = v_1v_4, \quad e_4 = v_2v_3, \quad e_5 = v_2v_4, \quad e_6 = v_3v_4,$$
$$e_7 = v_1v_2v_3, \quad e_8 = v_1v_2v_4, \quad e_9 = v_1v_3v_4, \quad e_{10} = v_2v_3v_4, \quad e_{11} = v_1v_2v_3v_4.$$

The mentioned $H_k$ is given by

$$H_0 = \emptyset, \quad H_1 = e_1e_2e_4, \quad H_2 = e_1e_2, \quad H_3 = e_1e_6e_7e_8e_9e_{10},$$
$$H_4 = e_{11}, \quad H_5 = e_1, \quad H_6 = e_1e_6, \quad H_7 = e_1e_2e_8e_{10},$$
$$H_8 = e_7e_8e_9e_{10}, \quad H_9 = e_8, \quad H_{10} = e_7, \quad H_{11} = e_1e_2e_4e_{11},$$
$$H_{12} = e_1e_2e_{11}, \quad H_{13} = e_1e_9e_{10}, \quad H_{14} = e_1e_2e_{10}, \quad H_{15} = e_3e_7,$$
$$H_{16} = e_1e_9, \quad H_{17} = e_1e_{11}, \quad H_{18} = e_1e_6e_{11}.$$

For example, $\Sigma H_0 = 0$ and $\Sigma H_1 = e_1 + e_2 + e_4 = v_1v_2 + v_1v_3 + v_2v_3$.

## 4. STS-graphs of Vasil'ev codes mod $J$

The dimension $\delta_k$ of $Ker(V_k)$ is, for $0 \leqslant k \leqslant 18$:

$$\delta_0 = 11; \quad \delta_i = 9 (i = 1, 2, 5); \quad \delta_i = 8 (i = 3, 9, 10); \quad \delta_k = 7 \text{ otherwise}.$$

The definition of each $V_k$ in Section 3 allows an ordered listing of its codewords as a sequence of 15-tuples $u_\ell$, $(1 \leqslant \ell \leqslant 2^{11})$, containing $J$ as its initial subsequence. If $\delta_k = 7$, then $J = K = Ker(V_k)$. If $\delta_k > 7$, $K$ is the union of $J$ and other $2^{11-\delta_k} - 1$ classes mod $J$; these special cases are considered in item G and in Subsection 4.2.

For $0 \leqslant k \leqslant 18$, the classes $\kappa_j$ of $V_k$ mod $J$, $(0 \leqslant j \leqslant 2^{11-\delta_k} - 1)$, are defined recursively by $\kappa_0 = J$ and $\kappa_j = \kappa_{j-1} + \tau_j$. Let $\kappa = \kappa_0, \kappa_1, \ldots, \kappa_{15}$ be the resulting order of the classes mod $J$. To obtain in practice the needed properties of $V_k$, a permuted order $\kappa' = \kappa'_0, \kappa'_1, \ldots, \kappa'_{15}$ from that of $\kappa$ was selected so that the new classes $\kappa'_0, \kappa'_1, \ldots, \kappa'_{15}$ are compatible with the use of the tensor product defined in item E.

Regardless of the representative $V_k$, $(0 \leqslant k \leqslant 18)$, the restrictions of the supports of $\tau_j$, $(j \in I_{15})$, to $I_{14} = I_{15} \setminus \{15\} \subset I_{15}$ are: (a) the complements in $I_7 \subset I_{15}$ of the successive triples of the ordered Steiner triple system STS(7) ={123, 145, 167, 246, 257, 347, 356}, (that is: taken from left to right, i.e. 4567, ..., 1247 $\subset I_{14}$, for $\tau_1, \ldots, \tau_7$, respectively; (b) the same seven triples taken backwards, i.e. 356, ..., 123 $\subset I_{14}$, for $\tau_8, \ldots, \tau_{14}$, respectively; and (c) $I_7 \subset I_{14}$, for $\tau_{15}$. The resulting sequence of classes equals $\kappa'$.

The codewords $\tau_j$ restrict to $LC_3$ over $I_7 \subset I_{14}$. Only their sets of 16 15th coordinates, $\{(\tau_j)_{15}; 0 \leqslant j \leqslant 15\}$, differ, for $0 \leqslant k \leqslant 18$. These are related via tensors, (see item H).

### 4.1. Properties of $H_J(V_k)$, $(k = 0, \ldots, 18)$

Let $\mu$ stand for the multiplicity of loops and links (that is non-looped edges) in the graphs $H_K(V_k)$ and $H_J(V_k)$. With the conditions given above, the following properties hold, for $0 \leqslant k \leqslant 18$:

(A) For each $v \in V_k$, the STS(15) $S(V_k, v)$ contains the triples $(t, t', 15)$, where $t \in I_7$ and $t' = 15 - t$. These seven triples (a) span $J$ as a linear subspace of $K$ and ((b) represent edges of $M(V_k)$ whose endvertices other than $v$ are codewords both in $v + J$ and in $v + L$. Moreover, $H_K(V_k)$ and $H_J(V_k)$) have loops of $\mu \geqslant 7$ and $\mu = 7$, respectively, at each vertex.

(B) Besides the seven triples in item A, each $S(V_k, v)$ contains the triples of exactly one V-fragment in each one of the following seven exclusively disjunctive pairs, where hexadecimal notation in $I_{15}$ is used due to our particular presentation preference:

(1) Either $F_1^+ = \{123, 1dc, e2c, ed3\}$ or $F_1^- = \{edc, e23, 1d3, 12c\}$;
(2) Either $F_2^+ = \{145, 1ba, e4a, eb5\}$ or $F_2^- = \{eba, e45, 1b5, 14a\}$;
(3) Either $F_3^+ = \{167, 198, e68, e97\}$ or $F_3^- = \{e98, e67, 197, 168\}$;
(4) Either $F_4^+ = \{246, 2b9, d49, db6\}$ or $F_4^- = \{db9, d46, 2b6, 249\}$;
(5) Either $F_5^+ = \{257, 2a8, d58, da7\}$ or $F_5^- = \{da8, d57, 2a7, 258\}$;
(6) Either $F_6^+ = \{347, 3b8, c48, cb7\}$ or $F_6^- = \{cb8, c47, 3b7, 348\}$;
(7) Either $F_7^+ = \{356, 3a9, c59, ca6\}$ or $F_7^- = \{ca9, c56, 3a6, 359\}$.

Each $F_m^+$ contains a triple of indices of linearly dependent rows of $H_7$, and each $t_1t_2t_3 \in F_m^+$ determines a corresponding $t'_1t'_2t'_3 \in F_m^-$. Furthermore, the triples in each of the resulting $V$-fragments associated to an $S(V_k, v)$ represent edges of $M(V_k)$ whose endvertices other than $v$ are codewords in a common class $w(v, F_m^{\varepsilon}) + J$, $(m \in I_7;\ \varepsilon = \varepsilon(m) = \pm)$. This guarantees that $H_K(V_k)$ and $H_J(V_k)$ have well-defined links, i.e. non-looped edges, with $1 \leqslant \frac{\mu}{4} \in \mathbb{Z}$ and $1 = \frac{\mu}{4} \in \mathbb{Z}$, respectively, at each vertex.

(C) As seen in Section 3, $H_J(V_k)$ can be viewed as the bipartite graph $K_{8,8} - F$ obtained from $K_{8,8}$ by deleting a 1-factor $F$. Let us represent each class $\kappa'_j$ by its index $j$, $(0 \leqslant j \leqslant 15 = f)$. Then we can denote the vertex parts of $H_J(V_k)$ by $P^+ = \{0, 1, \ldots, 7\}$ and $P^- = \{f, e, \ldots, 8\}$. Moreover,

$$F = \{0f, 1e, 2d, 3c, 4b, 5a, 69, 78\} = \{tt';\ 0 \leqslant t \leqslant 7\}.$$

For each $V_k$, $(0 \leqslant k \leqslant 18)$, we consider an adjacency list $A_k$ that, for each vertex of $P^+$ shows the respective vertices of $P^-$ and viceversa, adjacent by means of (the triples in) the V-fragments of item B, in the order in which they were presented. In fact, $A_k$ splits into sublists $A_k^+$ and $A_k^-$, showing the respective neighbors of the vertices of $P^+$ and $P^-$. From now on, we keep working only with $A_k^+$, as $H_J(V_k)$ is bipartite and $A_k^-$ behaves redundantly.

For instance, the $A_6^+$ and $A_6^-$ can be expressed as

$$\left|\begin{array}{l} 0:(+e,-d,-c,-b,-a,+9,-8);\\ 1:(-f,-c,-d,-a,-b,-8,+9);\\ 2:(-c,-f,+e,+9,-8,-b,-a);\\ 3:(+d,-e,+f,+8,-9,+a,+b);\\ 4:(-a,+9,-8,-f,+e,-d,-c);\\ 5:(+b,+8,-9,-e,+f,+c,+d);\\ 6:(+8,+b,+a,+d,+c,+f,-e);\\ 7:(-9,+a,+b,+c,+d,-e,+f). \end{array}\right| \quad \text{and} \quad \left|\begin{array}{l} f:(-1,-2,+3,-4,+5,+6,+7);\\ e:(+0,-3,+2,-5,+4,-7,-6);\\ d:(+3,-0,-1,+6,+7,-4,+5);\\ c:(-2,-1,-0,+7,+6,+5,-4);\\ b:(+5,+6,+7,-0,-1,-2,+3);\\ a:(-4,+7,+6,-1,-0,+3,-2);\\ 9:(-7,+4,-5,+2,-3,+0,+1);\\ 8:(+6,+5,-4,+3,-2,-1,-0), \end{array}\right|$$

respectively, where the sign $+$ or $-$ preceding the $m$th entry of the $p$th row ($m \in I_7$; $p = 0, \ldots, 7$ and $p = f, \ldots, 8$ respectively) indicates whether $F_m^+$ or $F_m^-$ is respectively in $S(V_k, v)$ for any $v$ in the class mod $J = Ker(V_6)$ represented by the vertex $p$ of $H_J(V_6) = H_K(V_6)$.

The signs in $A_k^+$ and $A_k^-$ can be displayed in a table $B_k$, with the signs along the diagonal, which are not determined by $A_k^+$ and $A_k^-$, defined so that the $B_k$'s are tensors as in item E. For example, for $k = 6, 14$:

| $B_6$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $f$ | − | − | − | + | − | + | + | + |
| $e$ | + | + | + | − | + | − | − | − |
| $d$ | − | − | − | + | − | + | + | + |
| $c$ | − | − | − | + | − | + | + | + |
| $b$ | − | − | − | + | − | + | + | + |
| $a$ | − | − | − | + | − | + | + | + |
| 9 | + | + | + | − | + | − | − | − |
| 8 | − | − | − | + | − | + | + | + |

| $B_{14}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $f$ | + | + | + | − | − | − | − | − |
| $e$ | − | − | − | + | + | + | + | + |
| $d$ | − | − | − | + | + | + | + | + |
| $c$ | − | − | − | + | + | + | + | + |
| $b$ | − | − | − | + | + | + | + | + |
| $a$ | − | − | − | + | + | + | + | + |
| 9 | − | − | − | + | + | + | + | + |
| 8 | − | − | − | + | + | + | + | +. |

Tables $B_k$ $(k = 0, \ldots, 18)$ are obtained using the same given technique.

(D) The list $A'$ obtained from $A_k^+$ by deleting the preceding signs is independent of $k = 0, \ldots, 18$. In fact, the triple formed by a row header number $i$ in $A$; a column order number $j$; and $t_{ij}$, where $t'_{ij} = 15 - t_{ij}$ is the entry in the $i$-row and $j$-column of any $A_k^+$, is either a triple of the STS(7) or $0aa$.

(E) Given two vectors $u = (u_f, \ldots, u_8)$ and $v = (v_0, \ldots, v_7)$ in $GF_2^8$, the *tensor* $u \otimes v$ of $u$ and $v$ is the $8 \times 8$-matrix $\{u_t v_s\}_{0 \leqslant t', s \leqslant 7}$, where $t' = 15 - t$. Then, the inner signed matrix $B'_k$ of $B_k$ is $u(k) \otimes v(k)$, so that $u(k) = (u_f, u_e, u_d, u_c, u_b, u_a, u_9, u_8)$ is the first column (or 0-column) of $B'_k$ and $v(k) = (v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ is the scalar product of the first coordinate $u_f(k)$ of $u(k)$ by the first row (or $f$-row) of $B'_k$. Note that the first coordinate of $v(k)$ is $v_0(k) = +$.

From now on, we replace the signs $+$ and $-$ in the expression of $u(k) \otimes v(k)$ by 0 and 1, respectively, (with the sign product rule replaced by the binary addition rule). For the examples of $B_k$ previous to item D, we have: $u(6) = 10111101$, $v(6) = 00010111$, $u(14) = 01111111$ and $v(14) = 00011111$.

We note that $H_J(V_k)$ depends on $u(k) \otimes v(k)$ in a unique manner up to change of representative in the class of $V_k$, so $H_J(V_k)$ is unchanged, up to the corresponding induced affine transformation in $Q_{15}$ that keeps $V_k$ invariant, and thus up to the resulting equivalence of STS(7)s and also up to the permutation of the set of classes $\{0, 1, \ldots, f\}$ having induced permutations on the set of vertex-parts $\{\{0, \ldots, 7\}, \{f, \ldots, 8\}\}$ of the bipartite graph $H_J(V_k)$ and on its set of non-edge pairs $F = \{0f, 1e, \ldots, 78\}$.

(F) If $K = Ker(V_k)$ has $\delta_k = 7$, then $H_K(V_k) = H_J(V_k)$ has associated tensor $u(k) \otimes v(k)$ and STS(15)-type associated to the classes $f, e, \ldots, 8$; $0, 1, \ldots, 7$, respectively, as follows, (where an additional rightmost column cites the orbit cardinality or size $|\mathcal{O}|$ of the code $V_k$, according to [10]):

| $k$ | $u(k) \otimes v(k)$ | $fedcba98$ | $01234567$ | $\|\mathcal{O}\|$ |
|---|---|---|---|---|
| 4 | $(01111111) \otimes (00000000)$ | 11111111 | 12222222 | 6 |
| 6 | $(10111101) \otimes (00010111)$ | 22222222 | 22222222 | 28 |
| 7 | $(11111111) \otimes (00011110)$ | 22222222 | 11111111 | 56 |
| 8 | $(10111111) \otimes (00010101)$ | 21222222 | 21222222 | 56 |
| 11 | $(10111111) \otimes (00111111)$ | 22222222 | 21222222 | 112 |
| 12 | $(01111111) \otimes (00001111)$ | 11111111 | 12222222 | 112 |
| 13 | $(00111111) \otimes (00010111)$ | 22222222 | 22222222 | 168 |
| 14 | $(01111111) \otimes (00011111)$ | 22212222 | 12222222 | 168 |
| 15 | $(10011111) \otimes (00000101)$ | 22222222 | 22222222 | 168 |
| 16 | $(10111111) \otimes (00000111)$ | 22221222 | 21222222 | 224 |
| 17 | $(10111111) \otimes (00000011)$ | 22222222 | 21222222 | 336 |
| 18 | $(00111101) \otimes (00010111)$ | 22222222 | 22222221 | 448 |

(G) For $k = 0, 1, 2, 3, 5, 9, 10$, we may use the notation for vertices of $H_K(V_k)$ in items C–E and proceed with the lists $A_k$, $A_k^+$ and $A_k^-$, the table $B_k$ and the tensor $u(k) \otimes v(k)$, provided some identifications are performed. Namely, symbols $f, e, \ldots, 8$ and $0, 1, \ldots, 7$ can be assigned respective values:

| $k$ | $\delta_k$ | $fedcba98$ | $01234567$ | $k$ | $\delta_k$ | $fedcba98$ | $01234567$ |
|---|---|---|---|---|---|---|---|
| 0 | 11 | 00000000 | 00000000 | | | | |
| 1 | 9 | 00332211 | 00332211 | 3 | 8 | 01675432 | 01675432 |
| 2 | 9 | 22221111 | 00003333 | 9 | 8 | 76543210 | 01234567 |
| 5 | 9 | 33221100 | 00112233 | 10 | 8 | 77442211 | 00335566 |

so the classes mod $K$ are split into classes mod $J$, where the 16 numbers shown in each case indicate which classes mod $J$, represented by the symbols $0, 1, \ldots, 7$; $f, e, \ldots, 8$, integrate in fact the classes mod $K$. Then, the table in item F can be completed with the following table:

| $k$ | $u(k) \otimes v(k)$ | $fedcba98$ | $01234567$ | $\|\mathcal{O}\|$ |
|---|---|---|---|---|
| 0 | $(00000000) \otimes (00000000)$ | 11111111 | 11111111 | 1 |
| 1 | $(00111111) \otimes (00111111)$ | 22222222 | 22222222 | 7 |
| 2 | $(11111111) \otimes (00001111)$ | 11111111 | 11111111 | 7 |
| 3 | $(11111101) \otimes (00000010)$ | 22222212 | 22222212 | 8 |
| 5 | $(00111111) \otimes (00000011)$ | 22222222 | 22222222 | 21 |
| 9 | $(01111111) \otimes (00000001)$ | 22222221 | 12222222 | 56 |
| 10 | $(00111111) \otimes (00111111)$ | 11111111 | 22222222 | 56 |

(H) Consider the codeword $\tau_j$ present in the four paragraphs previous to this subsection. The 15th coordinate $(\tau_j)_{15}$ of each codeword $\tau_j$ in $V_k$ will be denoted $\imath_j(k)$, for $j \in I_{15}$ and $k = 0, \ldots, 18$. Also, let $\imath_0(k) = 0$ be the 15th coordinate of the null codeword in $J$. It can be checked that these values $\imath_j(k)$ determine the tensors $u(k) \otimes v(k)$ defined in item E. More specifically, $u_{t'}(k) = \imath_{t'}(k)$ and $v_t(k) = \imath_t(k)$, for $0 \leqslant t \leqslant 7$.

The following theorem summarizes the facts presented in items A–H.

**Theorem 5.** *Let $J$ be the linear subspace of $V_k$, $(0 \leqslant k \leqslant 18)$, generated by the triples $(t, t', 15)$, $(t \in I_7)$. Then $H_J(V_k)$ is isomorphic to $K_{8,8} - F$ with vertex parts $P^+ = \{0, 1, \ldots, 7\}$ and $P^- = \{f, e, \ldots, 8\}$, and 1-factor $F = \{0f, 1e, 2d, 3c, 4b, 5a, 69, 78\}$. An adjacency list $A_k^+$ for the vertices of $P^+$ in terms of those of $P^-$ is obtained from the list $A'$ of item D, by prefixing to each one of its inner symbols, (namely $8, \ldots, f$), the corresponding sign from table $B_k$, whose inner signed matrix $u(k) \otimes v(k)$ is the tensor obtained from the 15th coordinates $\imath_j(k)$ of the codewords $\tau_j = (\alpha, 0000000, \imath_j(k)) \in \kappa'_j$, with $\alpha$ running on the linear $LC_3$, as in item H.*

### *4.2. Cases for which $H_K(V_k) \neq H_J(V_k)$*

For $k \in \{1, 2, 3, 5, 9, 10\}$, a modified adjacency list $\overline{A}_k$ of $H_K(V_k)$ can be obtained from the adjacency list $A_k$ by re-identifying the indices of the classes mod $J$ back to the indices of the classes mod $K$. This modified adjacency list $\overline{A}_k$ is again conceived at the level of the chosen order of the fragments $F_k^\varepsilon$. In what follows we present some properties of the graphs $H_K(V_k)$ and their adjacency lists $\overline{A}_k$, for $k \in \{1, 2, 3, 5, 9, 10\}$.

$H_K(V_1)$ is isomorphic to a complete graph $K_4$ on the vertex set $\{0, 1, 2, 3\}$ having uniform multiplicity $\mu = 8$, that is with its links having $\mu = 8$, and a loop of $\mu = 11$ at each vertex. This can be seen from $\overline{A}_1$, which is as follows:

$$\begin{array}{ll} 0 : (+0, -3, -3, -2, -2, -1, -1); & 2 : (+2, +1, +1, -0, -0, +3, +3); \\ 1 : (+1, +2, +2, +3, +3, -0, -0); & 3 : (+3, -0, -0, +1, +1, +2, +2). \end{array}$$

$H_K(V_2)$ is isomorphic to a 4-cycle on the vertex set $\{0, 1, 2, 3\}$ with a loop of $\mu = 7$ at each vertex and having two pairs of opposite edges: one pair with its two edges having $\mu = 12$, the other pair with its two edges having $\mu = 16$. This can be seen from $\overline{A}_2$, which is as follows:

$$\begin{array}{ll} 0 : (-2, -2, -2, -1, -1, -1, -1); & 2 : (-0, -0, -0, +3, +3, +3, +3); \\ 1 : (+3, +3, +3, -0, -0, -0, -0); & 3 : (+1, +1, +1, +2, +2, +2, +2). \end{array}$$

$H_K(V_3)$ is isomorphic to a complete graph $K_8$ on the vertex set $\{0, 1, \ldots, 7\}$ with uniform multiplicity $\mu = 4$ and a loop of multiplicity $\mu = 7$ at each vertex. This can be seen from $\overline{A}_3$, which is as follows:

$$\begin{array}{ll} 0 : (-1, -6, -7, -5, -4, +3, -2); & 4 : (-5, -2, +3, -1, -0, -7, -6); \\ 1 : (-0, -7, -6, -4, -5, -2, +3); & 5 : (-4, +3, -2, -0, -1, -6, -7); \\ 2 : (+3, -4, -5, -7, -6, -1, -0); & 6 : (-7, -0, -1, +3, -2, -5, -4); \\ 3 : (+2, +5, +4, +6, +7, +0, +1); & 7 : (-6, -1, -0, -2, +3, -4, -5). \end{array}$$

$H_K(V_5)$ is isomorphic to a complete graph $K_4$ on the vertex set $\{0, 1, 2, 3\}$ with a loop of multiplicity $\mu = 15$ at each vertex, and the links of $K_4$ forming three pairs of opposite links, two pairs with multiplicity $\mu = 8$ and the remaining pair with $\mu = 4$. This can be seen from $\overline{A}_5$, which is as follows:

$$\begin{array}{ll} 0:(+3,-2,-2,-1,-1,-0,-0); & 2:(-1,-0,-0,+3,+3,-2,-2); \\ 1:(-2,+3,+3,-0,-0,-1,-1); & 3:(+0,+1,+1,+2,+2,-3,-3). \end{array}$$

$H_K(V_9)$ is isomorphic to $K_8 - F'$ on the vertex set $\{0,\ldots,7\}$, where $F'$ is the 1-factor $\{07, 16, 25, 34\}$, with uniform multiplicity $\mu = 4$ and a loop of $\mu = 11$ at each vertex. This can be seen from $\overline{A}_9$, which is as follows:

$$\begin{array}{ll} 0:(-6,-5,-4,-3,-2,-1,-0); & 4:(-2,-1,-0,+7,-6,-5,-4); \\ 1:(+7,-4,-5,-2,-3,-0,-1); & 5:(-3,-0,-1,-6,+7,-4,-5); \\ 2:(-4,+7,-6,-1,-0,-3,-2); & 6:(-0,-3,-2,-5,-4,+7,-6); \\ 3:(-5,-6,+7,-0,-1,-2,-3); & 7:(+1,+2,+3,+4,+5,+6,-7). \end{array}$$

$H_K(V_{10})$ is isomorphic to the union of two link-disjoint bipartite graphs on common vertex parts $\{0, 3, 5, 6\}$ and $\{7, 4, 2, 1\}$ with common loops of $\mu = 7$ at each vertex. The two bipartite graphs here are: (a) the 1-factor $F' = \{07, 34, 52, 61\}$ with uniform multiplicity $\mu = 4$; (b) $K_{4,4} - F'$ with uniform multiplicity $\mu = 8$. This can be seen from $\overline{A}_{10}$, which is as follows:

$$\begin{array}{ll} 0:(+7,-4,-4,-2,-2,-1,-1); & 7:(+0,-3,-3,-5,-5,-6,-6); \\ 3:(+4,-7,-7,+1,+1,+2,+2); & 4:(+3,-0,-0,+6,+6,+5,+5); \\ 5:(+2,+1,+1,-7,-7,+4,+4); & 2:(+5,+6,+6,-0,-0,+3,+3); \\ 6:(+1,+2,+2,+4,+4,-7,-7); & 1:(+6,+5,+5,+3,+3,-0,-0). \end{array}$$

## 5. $H_K(C)$ is complete for Vasil'ev codes

The set of links of $H_K(V_k)$ can be split into 1-factors $\Phi_j$ according to the V-fragments they are labelled with. Because of the occurring identifications, these $\Phi_j$ have corresponding common multiplicities $\mu_j$ that extend to the null multiplicity if necessary. We take the multiplicities $\mu_j$ in the non-increasing order, for $j \in \{1,\ldots,\ell_k\}$, ($\ell_k = 2^{11}/2^{\delta_k}$).

Let us represent each $H_K(V_k)$ by the symbol $h_k = (\mu_0; \mu_1, \ldots, \mu_{\ell_k})$, where $\mu_0 \geqslant 7$ is the common multiplicity $\mu$ of the loops at the vertices of $H_K(V_k)$. Then, from what was said in Subsections 4.1 and 4.2, we obtain for $k > 0$:

$$\begin{array}{ll} h_1 = (11; 8, 8, 8), & h_3 = (7; 4, 4, 4, 4, 4, 4, 4), \\ h_2 = (7; 16, 12, 0), & h_9 = (11; 4, 4, 4, 4, 4, 4, 0), \\ h_5 = (15; 8, 8, 4), & h_{10} = (7; 8, 8, 8, 4, 0, 0, 0) \end{array}$$

and $h_k = (h_3, 0, 0, 0, 0, 0, 0, 0, 0)$, otherwise.

While the form $h_k$ of the STS-graph invariant $H_K(V_k)$ allows us to differentiate all Vasil'ev codes with $\delta_k > 7$, we can advance in the interpretation of $H_K(V)$ at the level of the associated labellings, as follows. As pointed out before item F, the definition of the tensors in item E can be given in terms of changes of coordinates. In our case, this involves permutations of coordinates and complementations of the signs + and − that appeared from item B on, or the corresponding replacing symbols 0 and 1. The pairs $(u_{i'}(k), v_i(k))$ formed by the $i$th coordinates of the vectors $u(k)$ and $v(k)$ can take values (0,0), (0,1), (1,0) and (1,1). Denote by $a_{ij}$ the number of times that $(i, j)$ appears as such a pair in $u(k) \otimes v(k)$, for

$i, j \in \{0, 1\}$. The matrix $A_k = (a_{ij})_{i,j=0,1}$ can be transformed by the mentioned changes of coordinates, amounting to at most a combination of the following actions: (a) permutation of the rows of $A_k$; (b) permutation of the columns of $A_k$; (c) transposition of $A_k$ via reflection along the main diagonal. Similarly, we can extract a likewise information from the columns in the table of item F that indicate the STS(15)-types 1 and 2 associated to the classes mod kernel: 0–7 and *f* to 8. The values of these types for classes 0, *f*; 1,*e*; 2,*d*; 3,*c*; 4,*b*; 5,*a* ;6,9; 7,8, form 2-tuples of the forms 11, 12, 21 and 22. For $V_k$, let $b_{ij}$ denote the number of such 2-tuples equal to $(i, j)$, where $i, j \in \{1, 2\}$. Then let $B = B_k = (b_{ij})_{i,j=1,2}$. We get:

$$
\begin{array}{llll}
A_4=\binom{10}{70}, & A_{12}=\binom{10}{34}, & & \text{where } B=\binom{10}{70};\\
A_6=\binom{11}{42}, & A_{13}=\binom{20}{24}, & A_{15}=\binom{20}{42}, & \text{where } B=\binom{00}{08};\\
A_{11}=\binom{10}{16}, & A_{17}=\binom{10}{52}, & A_{18}=\binom{21}{23}, & \text{where } B=\binom{01}{07};\\
A_{14}=\binom{10}{25}, & A_{16}=\binom{10}{43}, & & \text{where } B=\binom{01}{16};\\
A_7=\binom{00}{44}, & & & \text{where } B=\binom{08}{00};\\
A_8=\binom{10}{43}, & A_3=\binom{01}{70}, & & \text{where } B=\binom{10}{07},
\end{array}
$$

when $h_k = h_3$, completing a distinction between each two different classes of Vasil'ev codes, as claimed in Theorem 1.

## 6. Propelinear Vasil'ev codes

A propelinear code is necessarily homogeneous [3]. Thus, only $V_i$ may be nonlinear propelinear for $i = 1, 2, 5, 6, 13, 15$.

**Theorem 6.** *For* $1 \leqslant i \leqslant 18$, *the code* $V_i$ *is weakly propelinear*; *propelinear*; *and additive*, *if and only if* $i \in \{1, 2, 5, 6\}$; $i \in \{1, 2, 5\}$; *and* $i = 1$, *respectively*. *Moreover*, *the codewords* $u$ *in any class* $j \in V_i/K$ *have a common permutation* $p_j = \pi_u$, *where* $i = 1, 2, 5, 6$ *is fixed*. *Furthermore*, *the conditions of prelinearity and weakly propelinearity given in Section 1 combine into the following relation between classes* $j, k, \ldots$ *mod* $K$: *If* $u \in j$, *then* $v \in k$ *if and only if* $u * v = u + p_j(v) \in \ell$ *with* $p_\ell = p_j \circ' p_k$, *where* $j, k, \ell \in V_i/K$. *Defining an operation* $*'$ *on the set of classes mod K by means of* $j *' k = \ell$ *yields* $V_i/K$ *as a commutative group isomorphic to the group constituted by the permutations* $p_j$ *under* $\circ'$. *The group* $(V_i/K, *')$ *can be lifted to the group* $(V_i, *)$ [18] *if and only if* $i = 1, 2, 5$.

**Proof.** Let $x$ be a vertex of $H_K(V_k)$, where $0 \leqslant k \leqslant 18$. The $i$th parenthesized position $\varepsilon y$ in the row of $\overline{A}_k$ headed by $x$ is associated with $F_i^\varepsilon$, where $i \in I_7$ and $x, y$ are adjacent. It follows that every edge of $H_K(V_k)$ has associated either a V-fragment or a union of V-fragments, (together with $J$ in the case of a loop). Moreover, every edge $xy$ or $H_K(V_k)$ has associated a permutation of coordinates that allows to exchange the STS(15)'s of $x$ and $y$. In particular, the identity permutation, that we denote $p_0$, is associated to the loop at every vertex of $H_K(V_k)$. We need this information as detailed in the following paragraphs.

$\overline{A}_2$ implies that the links of $H_K(V_2)$ are 01, 23, 02, 13 and that they have associated the unions of fragments $F_4^- \cup F_5^- \cup F_6^- \cup F_7^-$, $F_4^+ \cup F_5^+ \cup F_6^+ \cup F_7^+$, $F_1^- \cup F_2^- \cup F_3^-$, $F_1^+ \cup F_2^+ \cup F_3^+$, respectively. It follows that the links of $H_K(V_2)$ with multiplicities $\mu=16$ and $\mu = 12$, namely the links in {01, 23} and {02, 13}, respectively, have associated coordinate permutation $p_1=(1,14)=(1e)$ and $p_2=\Pi_{2\leqslant t\leqslant 7}=(2d)(3c)(4b)(5a)(69)(78)$, respectively. Moreover, every vertex of $H_K(V_2)$ has a loop with associated set of triples $J$. Therefore, the vertices $v$ of every class $i$ have $p_i$ as their associated permutation $\pi_v$, for $i = 0, 1, 2, 3$, where $p_3 = p_1 \circ p_2$.

$\overline{A}_5$ and $\overline{A}_1$ imply that the component links of the 6-tuple (02, 13, 01, 23, 03, 12) in $H_K(V_5)$ and $H_K(V_1)$ have associated the corresponding components of ($F_2^- \cup F_3^-$, $F_2^+ \cup F_3^+$, $F_4^- \cup F_5^-$, $F_4^+ \cup F_5^+$, $F_1^+$, $F_1^-$) and ($F_4^- \cup F_5^-$, $F_4^+ \cup F_5^+$, $F_6^- \cup F_7^-$, $F_6^+ \cup F_7^+$, $F_2^- \cup F_3^-$, $F_2^+ \cup F_3^+$), respectively. It follows that the associated permutations are the corresponding components of $((2d), (2d), (1e), (1e), (1e)(2d), (1e)(2d)\}$ and $((1e)(3c), (1e)(3c), (1e)(2d), (1e)(2d), (2d)(3c), (2d)(3c))$, respectively. Moreover, every vertex of $H_K(V_5)$ and $H_K(V_1)$ has a loop with associated unions $J \cup F_6^- \cup F_7^-$ and $J \cup F_1^+$, respectively. Therefore, the vertices $v$ in every class $j$ have associated permutation $\pi_v = p_j$, where $(p_1, p_2, p_3)$ equals $((1e), (2d), (1e)(2d))$ and $((1e)(2d), (1e)(3c), (2d)(3c))$, respectively.

A 4×4-table relating $*'$ and $\circ$ is determined by $p_0 \circ p_j = p_j \circ p_0 = p_j$ and $p_j \circ p_j = p_0$, where $0 \leqslant j \leqslant 3$, and by $p_j \circ p_k = p_\ell$, where $1 \leqslant i, j, \ell \leqslant 3$, with $i \neq j \neq k$ and $i \neq k$. Using this table, weakly propelinearity can be verified for $V_2$, $V_5$ and $V_1$ for any selections of $\pi_u$ and $\pi_v$ among the permutations $p_j$, $(j = 0, 1, 2, 3)$. $V_6$ is not propelinear. However, a set $P_6$ of 16 adequate permutations for $V_6$ is given by $p_0 = g_0^+ = q_0 =$ identity and

$$\begin{array}{llll}
p_1 = g_3^- = q_c, & p_2 = g_5^+ = q_5, & p_3 = g_6^- = q_9, & p_4 = g_6^+ = q_6,\\
p_5 = g_5^- = q_a, & p_6 = g_3^+ = q_3, & p_7 = g_0^- = q_f, & p_8 = g_4^- = q_b,\\
p_9 = g_7^+ = q_7, & p_a = g_1^- = q_e, & p_b = g_2^+ = q_2, & p_c = g_2^- = q_d,\\
p_d = g_1^+ = q_1, & p_e = g_7^- = q_8, & p_f = g_4^+ = q_4, &
\end{array}$$

where $g_0^- = \Pi_{i=1}^7(ii')$, $g_i^- = (ii')$ and $g_i^+ = \Pi_{\ell\neq i}(\ell\ell')$, for $i = 1, \dots, 7$. Additional notation for permutations, namely $q_i = g_i^+$ and $q_{15-i} = g_i^-$, for $0 \leqslant i \leqslant 7$, allows to define an equivalence of projective spaces $PG(3, 2)$.

For $1 \leqslant j, k \leqslant 7$ and $\delta, \varepsilon = \pm$, define $g_j^\delta \circ' g_j^\varepsilon = g_0^{\delta\varepsilon}$, $g_j^\delta \circ' g_0^\varepsilon = g_0^\varepsilon \circ' g_j^\delta = g_j^{\delta\varepsilon}$ and $g_j^\delta \circ' g_k^\varepsilon = g_\ell^{\delta\varepsilon}$, where $jk\ell \in \{123, 145, 167, 246, 257, 347, 356\}$. Equivalently, $p_j \circ' p_k = p_\ell$ if and only if $jk\ell \in J \cup (\cup_{i=1}^7 F_i^+) \cup (\{0\ell\ell | \ell = 1, \dots, f\})$, that is, $jk\ell$ is either in the STS(15) of type 1 of the codewords of $V_0$ or of the form $0\ell\ell$. Thus, $V_6$ is weakly propelinear.

On the other hand, there is not a permutation $p_f$ that can be defined for $V_{13}$ or $V_{15}$, so these codes are not even prelinear.

It was shown in [4] that $V_1$ is additive by setting it in the form $\mathbb{Z}_2^7 \times \mathbb{Z}_4^4$. We reorder the 15 coordinates $01\dots f$ as $0e1d2c3b4a59678$, with the eight final coordinates taken as contiguous pairs yielding quaternary numbers by means of the Gray map $\phi : 00 \to 0,\ 01 \to 1,\ 11 \to 2,\ 10 \to 3$. Then the transpose parity check matrix $H$ whose rows are (0001111,0022), (0110011,0202) and (1010110,1111) on seven binary and four quaternary columns yields that a basis of $V_1$, namely $J \cup \{\tau_1, \tau_f, \tau_2, \tau_4, \tau_6\}$, is contained in $Ker(H)$, provided the scalar product of two vectors $A_i = (b_1^i, b_2^i, b_3^i, b_4^i, b_5^i, b_6^i, b_7^i, q_1^i, q_2^i, q_3^i, q_4^i)$, $(i = 1, 2)$, with $b_j^i \in \mathbb{Z}_2$ for $1 \leqslant i \leqslant 7$ and $q_j^i \in \mathbb{Z}_4$ for $1 \leqslant j \leqslant 4$, is given by $A_1 A_2 =$

$2(b_1^1b_1^2+\cdots+b_7^1b_7^2)+(q_1^1q_1^2+\cdots+q_4^1q_4^2)$, where operations within the first, (respectively second), pair of parentheses are binary, (respectively quaternary), and 2 times the binary result between the first two parentheses is interpreted in $\mathbb{Z}_4$, so $A_1A_2\in\mathbb{Z}_4$. However, neither $V_2$ nor $V_5$ is additive. □

## 7. Comparison between $H_K(C)$ and $H(C)$

For $0\leqslant k\leqslant 18$, a quotient graph $H(V_k)$ is obtained by identifying those vertices in $H_J(V_k)$ that have similar behavior with respect to the STS(15)-types 1 and 2. This graph, which is either a finite path of looped vertices or an isolated looped vertex, can be expressed in a bracketed expression with: (A) each quotient vertex $v$ indicated as $(\tau,\omega,\mu)$, where $\tau\in\{1,2\}\subset I_{80}$; $\omega=2^{-7}$ times the cardinality of the class of $v$; and $\mu=$ cardinality of the incident loop; (B) each quotient edge indicated as $(\tau_1,\omega_1,\mu_1)\beta_1,\beta_2(\tau_2,\omega_2,\mu_2)$, where $\beta_x=$ number of neighbors in $(\tau_y,\omega_y,\mu_y)$ from each representative of $(\tau_x,\omega_x,\mu_x)$ in $V_k$, $((x,y)\in\{(1,2),(2,1)\})$. Then, it can be seen from Section 4 that $H(V_k)$ is:

$$
\begin{aligned}
&[(1,16,35)]\quad \text{for } k=0,2;\\
&[(2,16,35)]\quad \text{for } k=1,3,5,6,13,15;\\
&[(1,8,7)28,28(2,8,7)]\quad \text{for } k=7,10;\\
&[(1,2,11)24,4(2,12,27)4,24(2,2,11)]\quad \text{for } k=9,14,16;\\
&[(1,1,7)28,4(1,7,7)24,24(2,7,7)4,28(1,1,7)]\quad \text{for } k=4,12;\\
&[(1,1,7)28,4(2,7,7)24,24(2,7,7)4,28(1,1,7)]\quad \text{for } k=8;\\
&[(1,1,7)28,4(2,7,7)24,24(2,7,7)4,28(2,1,7)]\quad \text{for } k=11,17,18.
\end{aligned}
$$

These are cases of the STS-graph $H(C)$ suggested in [6], whose formal definition is as follows. For each $v\in C$, write $S[v]=S(C,v)$ and use the same symbol $S[v]$ to denote the STS$(2^r-1)$-type number of $S(C,v)$, (assuming these numbers are chosen, which was our case for $n=2^r-1=15$, chosen as in [23,12,11]). With this second meaning for $S[v]$, let $T=\{w\in C;\,S[w]=S[v]\}$. Let $\mathscr{S}_0$ be the resulting partition of $C$ into these classes $T$. The symbol $T$ denoting these classes is used as well to denote $S[v]$ with any of its two given meanings, where $v\in T$.

To concretely define $H(C)$, a sequence $\mathscr{S}_0,\mathscr{S}_1,\ldots$ of partitions of $C$ starting with $\mathscr{S}_0$ is generated inductively. Assuming for a given integer $i\geqslant 0$ that $\mathscr{S}_i$ is already known, let $\rho_i(v,T)\leqslant n(n-1)/6$ be the cardinality of the set of neighbors $u$ of $v$ such that $S[u]=T$, for each $v\in C$ and $T\in\mathscr{S}_i$. Two vertices $v,w\in C$ are $\chi^{\mathscr{S}_i}$-equivalent if: (1) $S[v]=S[w]$ and (2) $\rho_i(v,T)=\rho_i(w,T)$, for every $T\in\mathscr{S}_i$.

If $v\in T\in\mathscr{S}_i$, let $T_i^v$ be the $|\mathscr{S}_i|$-tuple formed by the numbers $\rho_i(v,S)$, where $S$ varies in $\mathscr{S}_i$. If $T_i^v$ does not depend on the selection of $v\in T$, then we write $T_i=T_i^v$. Notice that the sum of the numbers in each $T_i$ is the regular degree $n(n-1)/6$ of $M(C)$.

For $i\geqslant 1$, the $i$th inductive step checks whether the $|\mathscr{S}_{i-1}|$-tuple $T_{i-1}^v$ associated to each $T\in\mathscr{S}_{i-1}$ and $v\in T$ does not depend on the selection of $v$. If there are no such dependencies, a quotient graph $H^{\mathscr{S}_{i-1}}(C)$ of $M(C)$ is well-defined, yielding the sought graph $H(C)$, since in that case the $\chi^{\mathscr{S}_{i-1}}$-equivalence classes yield well-defined $|\mathscr{S}_{i-1}|$-tuples $T_{i-1}$ of numbers $\rho_{i-1}(T,S)$ with $T\in\mathscr{S}_{i-1}$ fixed and $S$ varying in $\mathscr{S}_{i-1}$.

Otherwise: (a) let $\mathscr{S}_i$ be the refinement of $\mathscr{S}_{i-1}$ into maximal classes of vertices $v \in C$ with the tuples $T^v_{i-1}$ independent of the selection of $v$; (b) apply the $(i+1)$th inductive step. If $H(C)$ is determined totally in the $i$th inductive step, we say that $C$ has *inductive invariant* $\zeta(C) = i$.

The codes $C$ treated in [6], have $\zeta(C) \in \{1, 2\}$. Moreover, if $\zeta(C) = 1$ then the vertex set of $H(C)$ is $\mathscr{S}_0$; if $\zeta(C) > 1$ then a refinement of $\mathscr{S}_0$ yields the vertex set of $H(C)$. In fact, in [6] it was conjectured the assertion of Corollary 9.

**Proposition 7.** *$H(C)$ is a well-defined graph, for any 1-perfect code $C$.*

**Proof.** Because of the finiteness of $C$, we have that $H^{\mathscr{S}_{i-1}}(C)$ yields the desired $H(C)$, for some $i \geqslant 1$. □

**Corollary 8.** *The vertices of $H(C)$ are either classes or union of classes mod $K = Ker(C)$.*

**Proof.** Let $T$ be a vertex set of $H(C)$, thus representing a $\chi^{\mathscr{S}_{\zeta(C)}}$-equivalence class. We need to show that $K + x \subseteq T$, for every $x \in T$. If $y \in K + x$, then Lemma 2 yields $\mathscr{S}[y] = \mathscr{S}[x]$, which gives us $y \in T$. □

**Corollary 9.** *Every 1-perfect code $C$ has $\zeta(C) \in \{1, 2\}$.*

**Proof.** Departing from the partition $\mathscr{S}_0$ of $C$ into classes $T$, if the number $T^v_0$ does not depend on $v$, for each $T \in \mathscr{S}_0$ and each $v \in T$, then $\zeta(C) = 1$. Otherwise, there exists at least one $T \in \mathscr{S}_0$ with $v, w \in T$ such that $T^w_0 \neq T^v_0$. Let $\tau^v = Ker(C) + v \in C/Ker(C)$. Then, $\tau^w \cap \tau^v = \emptyset$, (but this does not imply that $T^w_0 \neq T^v_0$). Since $C$ is foldable, then: (a) $T^x_0 = T^w_0 \neq T^v_0$, for each $x \in \tau^w$ and (b) $T^y_0 = T^v_0 \neq T^w_0$, for each $y \in \tau^v$. Let $H_T$ be the set of classes $\tau^v \in C/Ker(C)$ such that $\tau^v \subseteq T$. Then $H_T$ has its partition into classes $\tau^v$ given in terms of the equivalence relation $\tau^v \sim \tau^u$ defined whenever $T^v_0 = T^u_0$. It is not difficult to see that $\mathscr{S}_1$ is a refinement of $\mathscr{S}_0$ such that $\tau^v$ does not depend on $v$, for each $T \in \mathscr{S}_1$ and each $v \in T$. Thus, $\mathscr{S}_1 = (\mathscr{S}_0 \backslash (\cup_{T \in \mathscr{S}_0}\{T\}) \cup (\cup_{T \in \mathscr{S}_0}(\cup_{\tau \in H_T/\sim}\{\tau\}))$ and $\zeta(C) = 2$. □

## 8. Open problems

The 16 vertices of each $H_J(V_k)$, $(0 \leqslant k \leqslant 18)$, and the 16 correspondingly associated classes of $V_k$ mod $J$ were denoted with the hexadecimal symbols $0, 1, \ldots, f = 15$ in a symmetric fashion related to the STS(7) of linearly dependent rows of the parity check matrix $H_7$, in the way prescribed by the seven exclusively disjunctive pairs of V-fragments in item B.

Is there any additional significance for the 1-1 correspondence between the 15 coordinates of codewords of $V_k$ and the classes $1, \ldots, f$ mod $J$?

Does class 0 have a particular additional meaning?

Possibly, this is related with the extended code $V'_k$ obtained from $V_k$ by adding a parity check 0th coordinate. By puncturing each one of the 16 coordinates of $Q_{16}$, the extended code $V'_k$ yields corresponding Vasil'ev codes which could be used to get a generalization of the STS-graph, that we call SQS-graph, in which vertices represent classes having similar local behavior for the Steiner quadruple systems (or SQS(16)s) on the 16 coordinate indices associated to the vertices of $V'_k$. (Recall that the minimum Hamming distance of an extended code of length 16 is 4, so that each vertex of $V'_k$ has associated a SQS(16), which contains 140 quadruples; there are 35 of these quadruples that have each one of the coordinate symbols $0, 1, \dots, 15$ fixed.)

In the same fashion, the 963 extended Phelps–Solov'eva codes of length 16 classified by Phelps in [14] offer a range of representation questions for SQS-graphs.

On the other hand, Rifà's 1-perfect code partitions [17] offer a ground for testing the behavior of these STS- and SQS-graphs. In particular, observe that we found above that codewords of Vasil'ev codes have associated only STS(15)-types 1 and 2.

Are there any non-Vasil'ev codes having only STS(15)-types 1 and 2?

It can be even questioned what additional 1-perfect codes $C$ of length 15 aside from the Vasil'ev codes $V_k$ have their edge labels expressible in terms of fragments, (or of length $\geqslant 15$).

Is there an extension of the results of Section 4 on Vasil'ev codes of length 15 to Vasil'ev codes of higher length by means of a technique using tensors generalizing those used above?

Do all (weakly) propelinear codes respect the relation between classes mod kernel via $*'$ and $\circ'$ of Theorem 6?

What other prelinear codes that are not (weakly) propelinear do exist?

Besides $V_6$, are there any other weakly propelinear codes which are not propelinear?

Recall that propelinear codes are homogeneous [4]. Are prelinear codes homogeneous?

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